Analytical Computation of Transverse Linearizations for Impulsive Mechanical Systems

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1 Hybrid Dynamical Model

Dynamics of a walking robot can be described by:

(i) A system of Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}(q,\dot q)}{\partial \dot q}\right) - \frac{\partial \mathcal{L}(q,\dot q)}{\partial q} = B(q)\, u, \qquad (1)$$

in the swing phase of motion, and

(ii) Updating laws modeling instantaneous impacts (i.e. with the walking surface), defined by a family of triples

$$\left\{\Gamma_-^{(i)},\, \Gamma_+^{(i)},\, F^{(i)}(\cdot)\right\}, \qquad i = 1, \ldots, N_d \qquad (2)$$

consisting of a collection of hyper-surfaces $\{\Gamma_-^{(i)},\, \Gamma_+^{(i)}\}$ in the state space of the mechanical system (1) and a family of mappings $F^{(i)}(\cdot)$ between pairs of these hyper-surfaces $F^{(i)} : \Gamma_-^{(i)} \to \Gamma_+^{(i)}$.

For simplicity, we will consider here only the case of $N_d = 1$ and drop the subindeces.

2 Virtual Holonomic Constraint

Every continuous-in-time sub-arc of a target periodic solution of (1), (2) – either in an open-loop or under a feedback action – defined by time-evolution of the generalized coordinates as:

$$q_1 = q_{1\star}(t), \quad \ldots, \quad q_n = q_{n\star}(t), \quad t \in [T_b, T_e]$$

can be alternatively described geometrically in the coordinate space as:

$$q_1 = \phi_1(\theta_\star), \quad \ldots, \quad q_n = \phi_n(\theta_\star), \quad \theta_\star \in [\Theta_b, \Theta_e],$$

where $\theta_\star$ could be some geometrical parameter such as the arc length along the pass or in many cases one can choose $\theta_\star$ to be one of the coordinates, e.g. $\theta_\star(t) = q_{n\star}(t)$ so that $\phi_n(\theta_\star) = \theta_\star$.

Theorem 1 *Projection of the dynamics of (1) onto a manifold Z, defined by invariance of n relations $q = \phi(\theta)$ in the case when $\dim u = \mathrm{rank}\,\{B(q)\} \le n-1$ can be described by*

$$\alpha(\theta)\ddot\theta + \beta(\theta)\dot\theta^2 + \gamma(\theta) = 0, \qquad (3)$$

where $\alpha(\theta)$, $\beta(\theta)$, and $\gamma(\theta)$ are scalar functions. The differential equation has a general integral of motion $I\Big(\theta(t), \dot\theta(t), \theta(0), \dot\theta(0)\Big)$, which keeps zero value along the solutions $\theta = \theta(t)$ and can be computed analytically. The part of the target trajectory $\theta = \theta_\star(t)$ is a solution of (3) with $\theta_\star(0) = \theta_0 \equiv \Theta_b$, $\dot\theta_\star(0) = \dot\theta_0$.

In some special situations it is possible to define a restriction of the whole hybrid dynamics onto the manifold Z. In general, this is not possible:

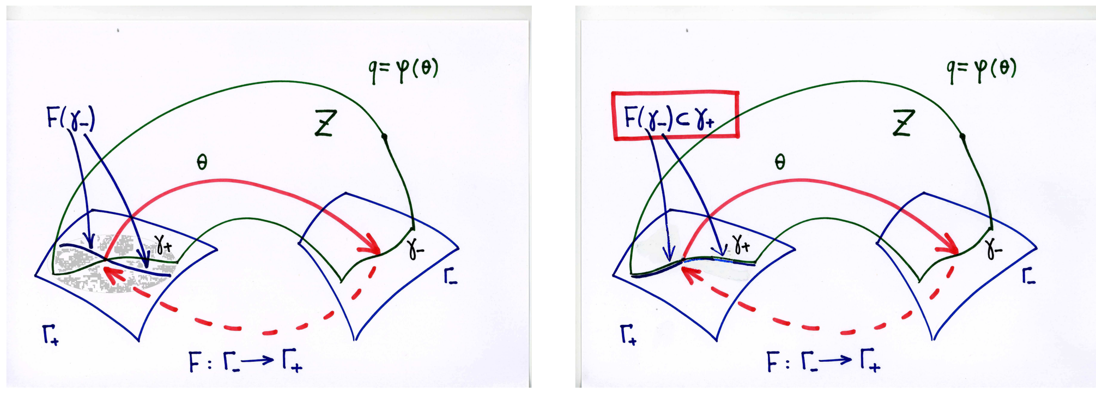

Figure 1: (Left): The discrete dynamics of the hybrid mechanical system do not necessary keep the 2-dimensional manifold Z invariant; (Right): For a class of mechanical systems the inclusion $F(\gamma_-) \subset \gamma_+$ holds allowing to define *hybrid zero dynamics*. The conditions are properties of the cycle and the model of the system and they are independent on a controller designed for stabilizing the periodic solution.

3 Transverse Linearization for Hybrid Cycle with One Jump

3.1 Linearizing continuous dynamics

The generic choice of $2n-1$ transversal coordinates $x_\perp = \Big[I\big(\theta, \dot\theta, \theta_\star(0), \dot\theta_\star(0)\big),\, y,\, \dot y\Big]^T$ allows to compute the linearization analytically:

- The 1st coordinate measures the distance to the target trajectory along the manifold Z.
- The other $2n-2$ coordinates

$$y_1 = q_1 - \phi_1(\theta), \quad \ldots, \quad y_{n-1} = q_1 - \phi_{n-1}(\theta),$$
$$\dot y_1 = \dot q_1 - \phi_1'(\theta)\,\dot\theta, \quad \ldots, \quad \dot y_{n-1} = \dot q_1 - \phi_{n-1}'(\theta)\,\dot\theta$$

measure distance to the manifold Z.

These coordinates define a moving Poincaré section $S(t)$: $t \in [0, T]$. The tangent hyper-planes to the sections in the beginning point ($t = 0$) and in the end point ($t = T$) of the continuous-in-time sub-arc typically do not coincide with the tangent hyper-planes to the switching hyper-surfaces:

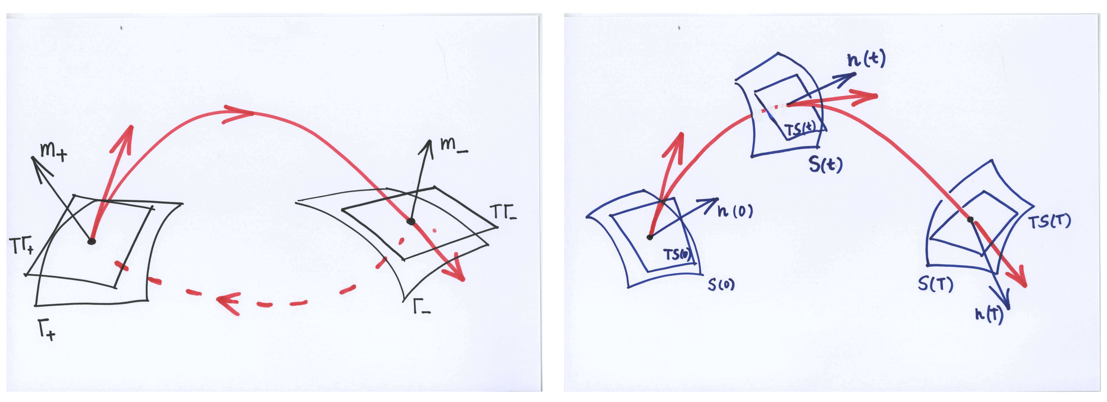

Figure 2: (Left): Tangent planes $T\Gamma_-$ and $T\Gamma_+$ of the switching surfaces Γ_- and Γ_+ at two points, where the periodic trajectory $q_\star(t)$ hits and originates from the switching surfaces. (Right): The moving Poincaré section – a family of $(2n-1)$-dimensional surfaces $S(t)$ transversal to the continuous-in-time sub-arc of the hybrid cycle. The linearization of transverse dynamicsis a linear control system defined on tangent planes $TS(t)$, $n(t)$ are normal to $TS(t)$

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A transverse linearization of (1) with $(n-1)$ independent controls u around $q_\star(t)$ is taken as

$$\frac{d}{dt}\zeta = \underbrace{\begin{bmatrix} a_{11}(t) & a_{12}(t) & a_{13}(t) \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix}}_{=A(t)} \zeta + \underbrace{\begin{bmatrix} b_1(t) \\ 0 \\ I \end{bmatrix}}_{=B(t)} w \qquad (4)$$

Here $\zeta \in \mathbb{R}^{2n-1}$ corresponds to $x_\perp$, $w \in \mathbb{R}^{n-1}$ is related to the part of u after partial feedback linearization; $a_{11}(\cdot)$, $a_{12}(\cdot)$, $a_{13}(\cdot)$, and $b_1(\cdot)$ are analytically computable functions on $[0, T]$ of appropriate dimensions.

3.2 Linearizing discrete dynamics

Linearization for the update law $\{\Gamma_+, \Gamma_-, F(\cdot)\}$ around the hybrid cycle $q_\star(t)$, is the differential of $F(\cdot)$ calculated at $[q_\star(T-), \dot q_\star(T-)] \in \Gamma_-$:

$$dF := \frac{\partial F}{\partial [q, \dot q]}\bigg|_{\substack{q = q_\star(T-) \\ \dot q = \dot q_\star(T-)}} : T\Gamma_- \to T\Gamma_+ \qquad (5)$$

where, $T\Gamma_-$, $T\Gamma_+$ are tangent subspaces to the C^1-smooth manifolds Γ_-, Γ_+ at two points where the periodic trajectory $q_\star(t)$ hits and originates from the switching surfaces.

3.3 Hybrid transverse linearization

To deal with the fact that $T\Gamma_- \ne TS(T)$ and $T\Gamma_+ \ne TS(0)$, we need to introduce two projection operators to match the two linearizations.

Let us denote by $P^+_{n(0)}$ the projection along $n(0)$ from $T\Gamma_+$ onto $TS(0)$ and by $P^-_{n(T)}$ be the projection along $n(T)$ from $TS(T)$ onto $T\Gamma_-$.

The analytically constructed transverse linearization is defined as follows.

For $t \in (kT, (k+1)T)$ the solution is defined by

$$\dot\zeta(t) = A(\tau)\,\zeta(t) + B(\tau)\,w(t), \qquad \tau = (t \mod T) \qquad (6)$$

where matrices $A(\cdot)$, $B(\cdot)$ are from (4). At $t = kT$, the state is instantaneously changed:

$$\zeta(t+) = \Big[P^+_{n(0)}\Big]\, dF\, \Big[P^-_{n(T)}\Big]\, \zeta(t-), \qquad (7)$$

4 Transverse Linearization for Orbital Stabilization

Theorem 2 *Suppose that there is a controller*

$$w(t) = K(\tau)\,\zeta(t), \qquad \tau = (t \mod T), \qquad (8)$$

that makes the equilibrium $\zeta = 0$ of the closed-loop system (6), (7), (8) exponentially stable.

Then, the feedback controller with partial feedback linearization and

$$v(t) = K\Big(\mathcal{T}([\theta(t), \dot\theta(t)])\Big) \begin{bmatrix} I(\cdot) \\ y(t) \\ \dot y(t) \end{bmatrix} \qquad (9)$$

makes the hybrid period motion $q_\star(t)$ of the hybrid mechanical system (1), (2) orbitally exponentially stable. $\mathcal{T}(\cdot)$ is any projection operator satisfying $\mathcal{T}([\theta_\star(t), \dot\theta_\star(t)]) = t$.